On the Generalized Spectrum of Linear Codes Associated to Grassmann Varieties and Schubert Varieties

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Linear Codes

$\mathbb{F}_q$: finite field with q elements.
$[n,k]_q$-code: a k-dimensional subspace of $\mathbb{F}_q^n$.
Hamming norm on $\mathbb{F}_q^n$:

$$\|x\| = |\{i : x_i \neq 0\}|, \quad x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n.$$

For a subspace D of $\mathbb{F}_q^n$,

$$\|D\| = |\{i : \exists\, x \in D \text{ such that } x_i \neq 0\}|.$$

Let C be an $[n,k]_q$-code.
C nondegenerate if $C \not\subseteq$ a coordinate hyperplane. Minimum distance of C:

$$d(C) = \min\{\|x\| : x \in C,\ x \neq 0\}.$$

rth higher weight of C:

$$d_r(C) = \min\{\|D\| : D \subseteq C,\ \dim D = r\}.$$

Weight Hierarchy of a Code

If C is a nondegenerate (linear) code of length n and dimension k, and if $d_r = d_r(C)$ denotes its rth higher weight, then we have:

$$0 = d_0 < d_1 < d_2 < \cdots < d_{k-1} < d_k = n.$$

The set $\{d_r : 0 \leq r \leq k\}$ is its weight hierarchy of the code C.

$\mathbb{P}^{k-1} :=$ $(k-1)$-dim projective space over $\mathbb{F}_q$.
$[n, k]_q-$ projective system X : A multi set of n points $\mathbb{P}^{k-1}$.
X is nondegenerate if $X \not\subseteq$ any hyperplane.

Theorem

[Tsfasmann-Vlăduț]: Equivalence classes of nondegenerate $[n, k]_q$-codes and nondegenerate $[n, k]_q$-projective systems are in $1-1$ correspondence.

If $C = C_X$ arises from a nondegenerate projective system $X \hookrightarrow \mathbb{P}^{k-1}$, with $|X| = n$, then the higher weights correspond to maximal linear sections of X. More precisely,

$d_r(C_X) = n - \max\{|X \cap E| : \mathrm{codim}\, E = r\}.E \subseteq \mathbb{P}^{k-1}, \mathrm{codim}\, E = r\}.$

In particular,

$$d(C_X) = n - \max\{|X \cap H| : H \text{ hyperplane}\}.$$

Generalized Spectrum of a Code

(usual) spectrum of $C :=$ the sequence $\{A_i(C)\}_{i=0}^{n}$, where, $A_i(C) :=$ the number of codewords of weight i in C. r-th

generalized spectrum of $C :=$ the sequence $\{A_i^r(C)\}_{i=0}^{n}, 0 \leq r \leq k$, where

$$A_i^r(C) := \#\{D : D \subseteq C, \dim D = r, ||D|| = i\}$$

r-th weight distribution $:= A^r(Z) = A_C^r(Z) = \sum_{i=0}^{n} A_i^r Z^i$. Note that $A^0(Z) = 1$ and $A_i^1 = \frac{A_i(C)}{(q-1)}$.

If Generalized spectrum of a code is known, then complete weight hierarchy is known!

Grassmann Varieties

V : vector space of dimension m over a field.
For $1 \leq \ell \leq m$, we have the Grassmann variety:

$$G_{\ell,m} = G_\ell(V) := \{\ell\text{-dimensional subspaces of } V\}.$$

Plücker embedding:

$$G_{\ell,m} \hookrightarrow \mathbb{P}^{k-1} \quad \text{where} \quad k := \binom{m}{\ell}.$$

Explicitly, the Plücker embedding is given as follows. Think of $\mathbb{P}^{k-1}$ as $\mathbb{P}(\bigwedge^\ell V)$. Given any $W \in G_\ell(V)$, choose a basis $\{w_1, \ldots, w_\ell\}$ of W. Identify W with $w_1 \wedge \cdots \wedge w_\ell$; the latter is uniquely determined up to multiplication by a nonzero constant.

In terms of coordinates, if we fix a basis $\{e_1, \ldots, e_m\}$ of V then $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_\ell}$ constitute a basis of $\bigwedge^\ell V$ as $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ varies over the indexing set $I(\ell, m)$ of all sequences $1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m$. If we write

$$w_1 \wedge \cdots \wedge w_\ell = \sum p_\alpha e_\alpha,$$

then (p_α) are the Plücker coordinates of W.
As a subset of $\mathbb{P}^{k-1}$, $G_{\ell,m}$ is given by the vanishing of certain quadratic homogeneous polynomials in the p_α's with integer coefficients. Thus $G_{\ell,m}$ is a projective algebraic variety. Also, $G_{\ell,m}$ is defined over $\mathbb{F}_q$.
The number of $\mathbb{F}_q$-rational points of $G_{\ell,m}$ is given by the Gaussian binomial coefficient

$$\begin{bmatrix} m \\ \ell \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{\ell-1})}{(q^\ell - 1)(q^\ell - q) \cdots (q^\ell - q^{\ell-1})}.$$

Grassmann Codes

Thanks to the Plücker embedding,

$$G_{\ell,m}(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k-1} \rightsquigarrow [n,k]_q\text{-code } C(\ell,m)$$

where the length n and the dimension k are:

$$n = \begin{bmatrix} m \\ \ell \end{bmatrix}_q \quad \text{and} \quad k = \binom{m}{\ell}.$$

Theorem [Ryan (1990), Nogin (1996)]:

$$d\left(C(\ell,m)\right) = q^{\delta} \quad \text{where} \quad \delta := \ell(m-\ell).$$

More generally, for $1 \leq r \leq \mu$ we have

$$d_r\left(C(\ell, m)\right) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1},$$

where $\mu := \max\{\ell, m-\ell\} + 1$.
[[Nogin (1996)];[Ghorpade-Lachaud(2000)]]

Thus the complete weight hierarchy of $C(2,4)$ is known!

Theorem [Hansen-Johnsen-Ranestad] On the other hand,

for $0 \leq r \leq \mu$,

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}).$$

e.g. If $\ell = 5, m = 10$, then $k = \binom{10}{5} = 252, \mu = 6$. We know the higher weights $d_1, d_2, \cdots, d_6$ and $d_{246}, \cdots, d_{252}$ due to above results. But, not complete weight hierarchy!

Problem 1: Determine all the higher weights d_r, $0 \leq r \leq k$, of the Grassmann code $C(\ell, m)$.

Schubert Codes

Let α be in $I(\ell, m)$, that is,

$$\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell, \; 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m.$$

Consider the corresponding Schubert variety

$$\Omega_\alpha := \{W \in G_{\ell,m} : \dim(W \cap A_{\alpha_i}) \geq i \; \forall i\},$$

where A_j is the span of the first j vectors in a fixed basis of our m-space. We have

$$\Omega_\alpha(\mathbb{F}_q) \hookrightarrow \mathbb{P}^{k_\alpha - 1} \rightsquigarrow [n_\alpha, k_\alpha]_q\text{-code } C_\alpha(\ell, m)$$

where

$$n_\alpha = |\Omega_\alpha(\mathbb{F}_q)| \text{ and } k_\alpha = |\{\beta \in I(\ell, m) : \beta \leq \alpha\}|,$$

with $\leq$ being the componentwise partial order.

Length of Schubert Codes

- If $\ell = 2$ and $\alpha = (m-h-1, m)$, then

  $n_\alpha = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)} - \sum_{j=1}^{h} \sum_{i=1}^{j} q^{2m-j-2-i}$ and

  $k_\alpha = \frac{m(m-1)}{2} - \frac{h(h+1)}{2}$. [Hao Chen (2000)]
- In general,

$$n_\alpha = \sum \prod_{i=0}^{\ell-1} \begin{bmatrix} \alpha_{i+1} - \alpha_i \\ k_{i+1} - k_i \end{bmatrix}_q q^{(\alpha_i - k_i)(k_{i+1}-k_i)}$$

  where the sum is over $(k_1, \ldots, k_{\ell-1}) \in \mathbb{Z}^\ell$ satisfying $i \leq k_i \leq \alpha_i$ and $k_i \leq k_{i+1}$ for $1 \leq i \leq \ell - 1$; by convention, $\alpha_0 = 0 = k_0$ and $k_\ell = \ell$. [Vincenti (2001)]

- $n_\alpha = \sum_{\beta \leq \alpha} q^{\delta_\beta}$, where $\delta_\beta = \sum_{i=1}^{\ell} (\beta_i - i)$.

[Ghorpade-Tsfasman (2005)]

Higher Weights of Schubert Codes

Proposition [Ghorpade-Lachaud(2000)]:

$$d\left(C_\alpha(\ell, m)\right) \leq q^{\delta_\alpha} \quad \text{where} \quad \delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i).$$

Minimum Distance Conjecture (MDC) [Ghorpade]:

$$\boxed{d\left(C_\alpha(\ell, m)\right) = q^{\delta_\alpha}.}$$

- True if $\alpha = (m - \ell + 1, \ldots, m - 1, m)$. [Nogin]
- True if $\ell = 2$. [Hao Chen (2000)];
  and independently, [Guerra-Vincenti (2002)].
- MDC is true, in general [Xu Xiang (2007)]

Only minimum weight is known!

Problem 2: Determine all the higher weights d_r, $0 \leq r \leq k$, of the Scubert Code code $C_\alpha(\ell, m)$.

Back to Higher Weights of Grassmann Codes

Recall: $\mu := \max\{\ell, m-\ell\}+1$ and

Theorem

[Nogin (1996)] and [Ghorpade-Lachaud (2000)]. For $1 \leq r \leq \mu$, we have

$$d_r\left(C(\ell, m)\right) = q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-r+1}.$$

Theorem

[Hansen-Johnsen-Ranestad] On the other hand, for $0 \leq r \leq \mu$,

$$d_{k-r}\left(C(\ell, m)\right) = n - (1 + q + \cdots + q^{r-1}).$$

These result cover several initial and terminal elements of the weight hierarchy of $C(\ell, m)$. Yet, a considerable gap remains.

Example:

$(\ell, m) = (2,5)$. Here $k = 10$, $\mu = 4$ and we know:

$$d_1, \ldots, d_4 \quad \text{as well as} \quad d_6, \ldots, d_{10}.$$

But d_5 seems to be unknown.

Example: $(\ell, m) = (2,6)$. Here $k = 15$, $\mu = 5$ and $d_6, \ldots, d_9$ are not known.

For $C(2,m)$ with $m \geq 2$, the values of d_r for $m \leq r < \binom{m-1}{2}$ do not seem to be known.

Theorem (Hansen-Johnsen-Ranestad)

$$d_5(C(2,5)) = q^6 + q^5 + 2q^4 + q^3 = d_4 + q^4.$$

Our First Progress

S.R.Ghorpade, A. R. Patil, Harish K. Pillai, *Decomposable subspaces, linear sections of Grassmann varieties, and higher weights of Grassmann codes*, **Finite Fields and Their Applications**, 15 (2009), 54-68.

Decomposable Subspaces and Structure Theorem:

- A vector $\omega \in \bigwedge^{\ell} V$ is said to be decomposable, if $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_\ell$ for some $v_1, v_2, \cdots, v_\ell \in V$.
- A subspace of $\bigwedge^{\ell} V$ is said to be decomposable, if all of its nonzero elements are decomposable.
- $V_\omega := \{v \in V : v \wedge \omega = 0\}$ for any $\omega \in \bigwedge^{\ell} V$.
- A Subspace E of $\bigwedge^{\ell} V$ of dimension r is said to be close of type I if there are $\ell + r - 1$ linearly independent vectors $f_1, \cdots, f_{\ell-1}, g_1, \cdots g_r$ such that $E = span\{f_1 \wedge \cdots \wedge f_{\ell-1} \wedge g_i : i = 1, 2, \cdots r\}$

- E is said to be close of type II if there exists $\ell+1$ linearly independent elements $u_1, \cdots, u_{\ell-r+1}, g_1, \cdots, g_r$ such that $E = span\{u_1 \wedge \cdots \wedge u_{\ell-r+1} \wedge g_1 \wedge \cdots \wedge \check{g}_i \wedge \cdots \wedge g_r : i = 1, \cdots, r\}$, where $\check{g}_i$ indicates g_i is deleted.
- Close subspace:= type I or Type II.

Structure Theorem E is decomposable $\iff$ E is closed.

Corollary: $\bigwedge^\ell V$ has a decomposable subspace of dimension r $\iff$ $r \leq \mu$, where $\mu = \max\{\ell, m-\ell\} + 1$.

- Structure Theorem+ Corollary gives d_r and d_{k-r} for $r \leq \mu$
- Structure Theorem + Corollary does not give d_r and d_{k-r} for $r > \mu$!

Lemma: Every $\mu + 1$ dimensional subspace of $\bigwedge^{\ell} V$ contains at most $q^{\mu} + q^3 - q^2 - 1$ decomposable vectors.

Lemma + Griesmer-Wei bound gives $d_{\mu+1}$ and $d_{k-\mu-1}$.

Complete weight hierarchy of $C(2,5)$ is known!

In the case of $C(2,6)$, we have $k = 15$ and $\mu = 5$. The previous results give us $d_1, \ldots, d_5$ and also $d_{10}, \ldots, d_{15}$. The above results yield the values of d_6 and d_9. But d_7 and d_8 are not covered.

We have to do something more!

Our Second Progress

S.R.Ghorpade, A. R. Patil, Harish K. Pillai, *Subclose families, threshold graphs, and the weight hierarchy of Grassmann and Schubert Codes*, **Contemporary Mathematics, American Mathematical Society**, Vol. 487 (2009), 87-99.

Given any family $\Lambda = \{\alpha^{(1)}, \ldots, \alpha^{(r)}\}$ of ℓ-subsets of $\{1, \ldots, m\}$, let

$$K_\Lambda := \sum_{1 \le i < j \le r} |\alpha^{(i)} \cap \alpha^{(j)}|.$$

For $0 \le r \le k$, define

$$K_r := \max \{K_\Lambda : \Lambda \subseteq I(\ell, m), \ |\Lambda| = r\}.$$

Given any $\Lambda \subseteq I(\ell, m)$ with $|\Lambda| = r$, we say that Λ ie a subclose family if $K_\Lambda = K_r$.

Observe that a close family is subclose. However, subclose families of cardinality r exist for every $r \leq k := \binom{m}{\ell}$.
Fact: Given any $\Lambda \subseteq I(\ell, m)$, if we let $\Lambda^c := I(\ell, m) \setminus \Lambda$, then

$$\Lambda \text{ is subclose } \iff \Lambda^c \text{ is subclose.}$$

When $r > \mu$, there is no close family of cardinality r and this is partly a reason why the methods discussed above do not extend in the general case. Using the notion of Subclose family, we give the following conjecturs.

Conjecture

For any positive integer $r \leq k$, the higher weights d_r of $C(\ell, m)$ are attained by linear sections of $G(\ell, m)$ by projective linear subspaces corresponding to subclose families of $I(\ell, m)$.

It may be noted, however, that for general r, the sections of $G(\ell, m)$ by subspaces Π_Λ corresponding to subclose families Λ may be of varying cardinality, and it is necessary to choose a maximal family in order to determine the higher weight. Thus, the above conjecture narrows the search for a likely value of d_r but gives no specific information in general. With this in view, we propose the following explicit conjectural formula in certain special cases.

Conjecture

Let $\mu = \max\{\ell, m-\ell\} + 1$ and $\delta = \ell(m-\ell)$. Then the $(\mu + j)$th higher weight of the Grassmann code $C(\ell, m)$ is given by

$$d_{\mu+j} = d_{\mu+j-1} + q^{\delta-(j+1)} \qquad \text{for } j = 1, 2, \ldots, m-\ell.$$

Consequently, for $j = 1, 2, \ldots, m-\ell$, we have

$$d_{\mu+j} = \left(q^{\delta} + q^{\delta-1} + \cdots + q^{\delta-\mu+1}\right) + \left(q^{\delta-2} + q^{\delta-3} + \cdots + q^{\delta-(j+1)}\right).$$

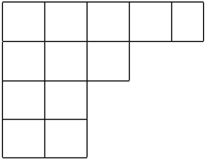
Complete weight hierarchy of $C(2,6)$ and $C(2,7)$ is known!

Our Third Progress

S.R.Ghorpade, T. Johansen, A. R. Patil, Harish K. Pillai, *Higher weights of Grassmann codes in terms of properties of Schubert unions*, **arXive:1105.0087v1 [math.AG], 30 April, 2011**.

In this paper we give an explicite algorithm to determine the complete weight hierarchy of $C(2, m)$ using so called Young tableaux and Schubert unions.
For any $k \in \mathbb{Z}^+$, its partition:= a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ such that $k = \lambda_1 + \lambda_2 + \cdots + \lambda_t = |\lambda|$.
The set $\{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq t,\ 1 \leq j \leq \lambda_i\}$ is called the *Ferrer's diagram or Young diagram of shape λ* where (i, j) is the *cell* in row i and column j.

A partition can be described by its Young diagram (*Ferrer's diagram*) which consists of t rows, with the first row containing λ_1 boxes, the second row containing λ_2 boxes, etc. For example, $\lambda = (5, 3, 2, 2)$ is drawn as follows.



Area of Young Tableaux: $=$ Number of boxes (i.e. k)
Subtableaux:$=$ For a number $r < k$, a tableaux T with partition $\chi = (\chi_1, \ldots, \chi_t)$ if $\chi_1 \leq \lambda_1, \chi_2 \leq \lambda_2, \ldots, \chi_t \leq \lambda_t$ and $\chi_1 + \chi_2 + \ldots + \chi_t = r$.

Recipe for complete weight hierarchy of $C(2, m)$

- Associate to $C\,(2, m)$ a specific strict Young tableaux Y with a special filling: $m-1$ boxes in first row, $m-2$ boxes in the second row, and so on down to one box in the $(m-1)$th row.
- For fixed $r, 1 \leq r \leq k$, choose a family of strict subtableux $\mathcal{F} = \{T_1, T_2, \dots, T_p\}$ with area r.
- Find a_{ih}:= The number of times that i occurs in the hth subtableaux $T_h \in \mathcal{F}$.
- Associate $\gamma_h = \sum_i a_{ih} q^i$ to the subtableaux T_h for each h.
- Find $g^r(2, m) = \max_h \gamma_h$.
- $d_{k-r} = n - g^r(2, m)$

Complete weight hierarchy of $C(2,6)$

Here $m = 6$, $k = 15$, $\mu = 5$ and $\delta = 8$. We have the following diagram. (filling explained as above)

0	1	2	3	4
2	3	4	5	
4	5	6		
6	7			
8				

e.g. Let r=4. We have two strict subtableau,

$T_1 = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline\end{array}$, and $T_2 = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$. We have,

$\gamma_1 = 1 + q + q^2 + q^3$ and $\gamma_2 = 1 + q + 2q^2$. Hence, we have $g^4(2,6) = \max\{\gamma_1, \gamma_2\} = \gamma_1 = 1 + q + q^2 + q^3$. Therefore,

$d_{11} = d_{15-4} = n - \gamma_1 = n - (1 + q + q^2) =$
$q^9 + 2q^8 + 3q^7 + 4q^6 + 5q^5 + 4q^4 + 5q^3 + 2q^2 + q.$

Cont..

r	$g^r(2,m)$	$d_{k-r} = n - g^r(2,m)$
0	0	n
1	1	$n-1$
2	$1+q$	$n-(1+q)$
3	$1+q+q^2$	$n-(1+q+q^2)$
4	$1+q+q^2+q^3$	$n-(1+q+q^2+q^3)$
5	$1+q+q^2+q^3+q^4$	$n-(1+q+q^2+q^3+q^4)$
6	$1+q+2q^2+q^3+q^4$	$q^3+2q^4+2q^5+2q^6+q^7+q^8$
7	$1+q+2q^2+2q^3+q^4$	$2q^4+2q^5+2q^6+q^7+q^8$
8	$1+q+2q^2+2q^3+2q^4$	$q^4+2q^5+2q^6+q^7+q^8$
9	$1+q+2q^2+2q^3+2q^4+q^5$	$q^4+q^5+2q^6+q^7+q^8$
10	$1+q+2q^2+2q^3+2q^4+q^5+q^6$	$q^4+q^5+q^6+q^7+q^8$
11	$1+q+2q^2+2q^3+3q^4+q^5+q^6$	$q^5+q^6+q^7+q^8$
12	$1+q+2q^2+2q^3+3q^4+2q^5+q^6$	$q^6+q^7+q^8$
13	$1+q+2q^2+2q^3+3q^4+2q^5+2q^6$	q^7+q^8
14	$1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7$	q^8
15	$1+q+2q^2+2q^3+3q^4+2q^5+2q^6+q^7+q^8$	0

Back to Generalized Spectrum

Recall that (usual) spectrum of $C :=$ the sequence $\{A_i(C)\}_{i=0}^n$, where, $A_i(C) :=$ the number of codewords of weight i in C.
r-th generalized spectrum of $C :=$ the sequence $\{A_i^r(C)\}_{i=0}^n, 0 \leq r \leq k$, where

$$A_i^r(C) := \#\{D : D \subseteq C, \dim D = r, ||D|| = i\}$$

- Nogin gave usual spectrum of $C\,(2, m)$.
- Vincenti-Montanucci computed rth generalized spectrum of $C(2, 4)$ using the work of Tallini (1974) for some r.

Here we complete and correct their calculations and give an alternative method to compute the generalized spectrum of $C(2, 4)$ (In progress for publication).

Theorem The generalized spectrum $\{A_i^r\}$ $0 \le r \le k$ $0 \le i \le n$ of $C(2,4)$ is:

- $r = 1.$ $$A_{q^4}{}^1 = A^1_{d_1} = q^4 + q^3 + 2q^2 + q + 1;$$ $$A^1_{q^4+q^2} = q^5 - q^2; \text{ and } A_i{}^1 = 0, \text{ elsewhere.}$$
- $r = 2$ $$A^2_{q^4+q^3} = A_{d_2}{}^2 = (q^3 + q^2 + q + 1)(q^2 + q + 1);$$ $$A^2_{q^4+q^3+q^2-q} = \frac{q^4}{2(q^4+q^3+2q^2+q+1)};$$ $$A^2_{q^4+q^3+q^2} = A^2_{d_3} = (q^3 - q^3)(q^2+1)(q^2+q+1);$$ $$A^2_{q^4+q^3+q^2+q} = \frac{q^4}{2(q^3-1)(q-1)};$$ $$A^2_i = 0; \quad \text{elsewhere.}$$
- $r = 3$ $$A^3_{q^4+q^3+q^2} = A_{d_3}{}^3 = 2(q^3 + q^2 + q + 1);$$ $$A^3_{q^4+q^3+2q^2-q} = \frac{q^2(q+1)^2(q^2+1)(q^2+q+1)}{2};$$ $$A^3_{n-(q+1)} = q^4(q^3-1)(q^2+1)+(q^4-1)(q^2+q+1);$$ $$A^3_{n-1} = \frac{q^2(q^4+1)(q^2-q+1)-2q^3}{2};$$ $$A^3_i = 0; \quad \text{elsewhere.}$$

Theorem (cont...)

- $r = 4$

$$\begin{aligned} A^4_{q^4+q^3+2q^2} &= A^4{}_{d_4} = (q^2+1)(q^2+q+1)(q+1); \\ A^4_{(n-2)} &= \frac{q^4(q^2+1)(q^2+q+1)}{2}; \\ A^4_{(n-1)} &= q(q^4+1)(q^4+q+1)(q-1); \\ A^4_{n-0} &= \frac{q^4(q-1)(q^3-1)}{2}; \\ A^4_i &= 0; \quad \text{elsewhere.} \end{aligned}$$

- $r = 5$

$$A^5_{q^4+q^3+2q^2+q} = A^4{}_{d_5} = (q^2+1)(q^2+q+1);$$

$$A^5_i = 0; \quad \text{elsewhere.}$$

- $r = 6$

$$\begin{aligned} A^6_{q^4+q^3+2q^2+q+1} &= 1; \\ A^6_i &= 0; \quad \text{elsewhere.} \end{aligned}$$

Outline of the Proof

A quadric Q_N in $\mathbb{P}^N := V(F)$, where

$$F = \sum_{i=0}^{N} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j$$

F nondegenerate $:=$ If $F \not\sim$ a form in less than $(N+1)$ variables. Q_N nonsingular $:=$ If F is nondegenerate.
For N even, $Q_N \sim P_N = V(x_0^2 + x_1x_2 + \ldots + x_{N-1}x_N)$,
For N odd, $Q_N \sim H_N = V(x_0x_1 + x_2x_3 + \ldots + x_{N-1}x_M)$,or
$Q_N \sim E_N = V(f(x_0, x_1) + x_2x_3 + \ldots + x_{N-1}x_N)$,
$f(x_0, x_1) =$ irreducible quadratic form over $\mathbb{F}_q$.
Generator $:=$ Subspace of maximum dimension on W_N (general quadric). Projective Index $(g) :=$ Dimension of a generator. Character $(w) := w(Q_N) = 2g - N + 3$.

Number of sections of nonsingular quadrics

$S(m,t,v;N,w) := \{\Pi_m : \Pi_m \cap Q_N = \Pi_{m-t-1} Q_t\}$. where $w(Q_N) = w(Q_t)$ $N(m,t,v;N,w) := \#S(m,t,v;N,w)$.

Proposition 1 [Hirschfeld, Thas] [Chapter 22, General Galois Geometry]:

$$N(m,t,v;N,w) = q^{\frac{1}{2}\left\{T[t+1+vw(2-v)(2-w)]-v(2-v)(w-1)^2\right\}} \times$$
$$\left[\frac{1}{2}\left\{T+v+(1+3v-2v^2)w-v(2-v)w^2\right\}, \frac{1}{2}(N+1-w)\right]_+ \times$$
$$\left[\frac{1}{2}\left\{T+2-v-(1-5v+2v^2)w-v(2-v)w^2\right\}, \frac{1}{2}(N-1+w)\right]_-$$
$$\left\{\left[v(2-v), \frac{1}{2}(t+1-v)\right]_+ \left[1, \frac{1}{2}(t-1+v)\right][1, m-t]_-\right\},$$

where

$$T = N + t - 2m$$

Theorem 2 [Nogin]

The generalized spectrum $\{A_i^r\}$ of the code C associated with Q_{k-1} in $\mathbb{P}^{k-1}$ is given by

- $A_i^r = N(k-r-1, t, v; k-1, w)$
  if $i = n - \tau(k-r-t-2, t, v)$ for some t, v such that
  $k-r-1 \geq t \geq \max\{k-2r-1+|w-v|, 1-v\}$,
  $$v = 0 \textit{ or } 2, t \equiv 1 (\mathrm{mod} 2)$$
- $A_i^r = \sum_t N(k-r-1, t, v; k-1, w)$,
  if $i = n - \theta_{k-r-2}$, where t runs over odd integers such that
  $$k-r-1 \geq t \geq \max\{k-2r-1+|w-v|, 0\}$$
- $A_i^r = 0$, elsewhere.

Observation:

- $G(2,4)$ is the hyperbolic quadric (Klien quadric) in $\mathbb{P}^5$.
- $g = 2, w = 2$
- $C(2,4)$ has
  $n = |Q_5(\mathbb{F}_q)| = \theta_4 + q^2 = q^4 + q^3 + 2q^2 + q + 1$ and
  $k = 6$.

Theorem 1 + Theorem 2 + The observation =complete weight hierarchy of $C(2,4)$

e.g. For $r = 3, v = 2, t = 1$, we have
$i = n - (2q+1) = q^4 + q^3 + 2q^2 - q$ and

$$A^3_{q^4+q^3+2q^2-q} = \frac{q^2}{2}(q^2+1)(q+1)^2(q^2+q+1).$$

Q.E.D.

Future Plan and scope to collaborate

- Complete weight hierrachy of $C(\ell, m)$
- Complete weight hierrachy of Schubert codes
- Generalized spectrum of $C(2, m)$ and then $C(\ell, m)$
- Usual spectrum of Schubert codes

Thank You